**Math 122** **Name:** Solutions

**Joseph C Foster** **March 26th, 2018**

**Spring 2018** **Time Limit: 50 minutes**

**Exam 2.5**

This exam contains 8 pages (including this cover page) and 15 questions.
The total number of marks is 150. You have 50 minutes to complete the exam.

Read each question carefully. When specified, you must show **all** *necessary* work to receive full credit.

No super fancy calculators/phone/smartwatch allowed under any circumstances. Place these items in your bag, out of reach. Cheating of any kind will not be tolerated and will result in a grade of zero. A regular calculator is allowed.

| Question | Marks | Score | Question | Marks | Score |
|---|---|---|---|---|---|
| 1 | 5 | | 9 | 5 | |
| 2 | 5 | | 10 | 5 | |
| 3 | 5 | | 11 | 20 | |
| 4 | 5 | | 12 | 20 | |
| 5 | 5 | | 13 | 20 | |
| 6 | 5 | | 14 | 20 | |
| 7 | 5 | | 15 | 20 | |
| 8 | 5 | | Total | 150 | |

| Total Possible | Your Current Score | Your Current Percentage | Your Current Grade |
|---|---|---|---|
| 570 | | | |

For questions 1-3, determine if the statement is true or false. By true we mean *always* true.

1. (5 marks) If $x = a$ is a stationary point of $f(x)$, then $f'(a) = 0$.

   A. True B. False

2. (5 marks) If $f(x)$ is defined on the interval $a \leq x \leq b$, then the global extrema of $f(x)$ on $[a, b]$ occur only at the critical points of $f(x)$ on $[a, b]$.

   A. True B. False

3. (5 marks) The tangent line to the function $f(x) = 3x^2 - 5$ at the point $(2, 5)$ is $y = 12x - 19$.

   A. True B. False

For questions 4-7, fill in the blanks.

4. (5 marks) If $f'(a) = 0$ and $f''(a) > 0$, then $x = a$ is a local minimum of $f(x)$.

5. (5 marks) If $f(x)$ changes concavity at $x = a$, then we call $a$ an inflection point of $f(x)$.

6. (5 marks) The function $f(x) = x^3 - e^x$ is concave down at $x = 0$ and concave up at $x = 1$.

   $f'(x) = 3x^2 - e^x$

   $f''(x) = 6x - e^x$

   $f''(0) = -1 < 0$

   $f''(1) = 6 - e > 0$

7. (5 marks) Complete the table below.

| | |
|---|---|
| $\frac{d}{dx}(c) = 0$ | $\frac{d}{dx}(x^n) = nx^{n-1}$ |
| $\frac{d}{dx}(a^x) = \ln(a)a^x$ | $\frac{d}{dx}(\ln(x)) = 1/x$ |

For questions 8-10, choose the best answer. There is only one correct answer but you may choose up to *two*. If you choose two and one of the answers is correct, you will receive 3 marks.

8. (5 marks) Suppose $f(8) = -3$, $f'(8) = 2$, $g(8) = -18$ and $g'(8) = 4$. What is $\frac{d}{dx}f(x)g(x)$ at $x = 8$?

A. $-48$

B. $-12$

C. 83

D. 92

9. (5 marks) Which of the following would be the best way to maximise profit?

A. Minimise revenue and minimise cost

B. Minimise revenue and maximise cost

C. Maximise revenue and minimise cost

D. Maximise revenue and maximise cost

10. (5 marks) Which of the following could be the function $f(x)$ with $f'(x) = 3x^2 + 3e^{3x}$?

A. $f(x) = x^3 - e^{-3x} + 2$

B. $f(x) = 3x^3 + 3e^{3x} - 1$

C. $f(x) = x^2e^{-3x}$

D. $f(x) = x^3 + e^{3x} - 1$

For questions 11-15, show **all** *necessary* work to receive full credit. Please circle or box your final answer. If you cannot complete a problem but can write down what you want to do, and this is correct, you can still receive partial credit. Don't leave anything blank!

11. (a) (4 marks) Differentiate $y = 3x - x^2$, with respect to $x$.

$y' = \boxed{3 - 2x}$

(b) (6 marks) Differentiate $y = (5 - x)^{-1}$, with respect to $x$.

Chain Rule

$f(x) = x^{-1}$ $g(x) = 5 - x$

$f'(x) = -x^{-2}$ $g'(x) = -1$

$y = f(g(x))$

$y' = g'(x) f'(g(x))$

$= -(-(5-x)^{-2})$

$= \boxed{(5-x)^{-2}}$

(c) (6 marks) Differentiate $f(x) = \dfrac{3x - x^2}{5 - x}$, with respect to $x$.

$= (3x - x^2)(5-x)^{-1}$ Product Rule

$u(x) = 3x - x^2$

$u'(x) = 3 - 2x$

$v(x) = (5-x)^{-1}$

$v'(x) = (5-x)^{-2}$

$f(x) = u(x)v(x)$

$f'(x) = u'(x)v(x) + u(x)v'(x)$

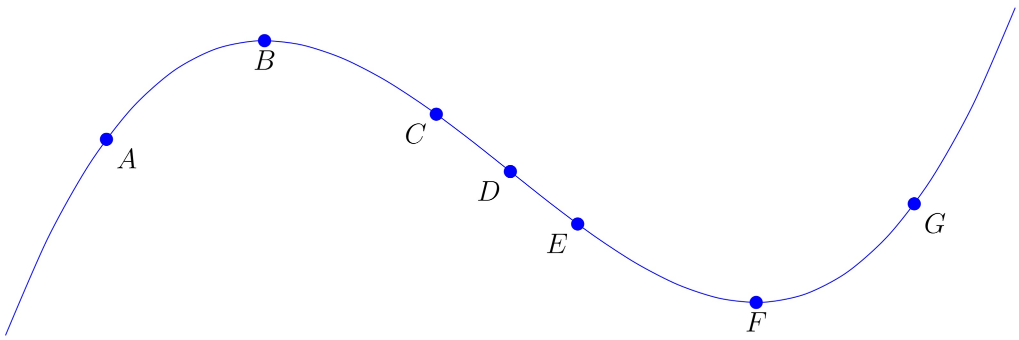
$= \boxed{(3-2x)(5-x)^{-1} + (3x - x^2)(5-x)^{-2}}$

(d) (4 marks) Is $f(x)$ increasing, decreasing or stationary at $x = 2$? (circle *one*)

Increasing Decreasing Stationary

$(3-4)(5-2)^{-1} + (6-4)(5-2)^{-2} = -\frac{1}{3} + \frac{2}{9} = -\frac{1}{3} < 0$

12.




Consider the graph of the function $f(x)$ above.

(a) (10 marks) Determine if each of the following quantities are positive, negative or zero. (circle *one*)

i. The derivative at $C$

Positive Negative Zero

ii. The second derivative at $C$

Positive Negative Zero

iii. The second derivative at $D$

Positive Negative Zero

(b) (10 marks) Find the *single* point that matches the following descriptions. (circle *one*)

i. $f'(x) > 0,\ f''(x) > 0$

$A$ $B$ $C$ $D$ $E$ $F$ $G$

ii. $f'(x) > 0,\ f''(x) < 0$

$A$ $B$ $C$ $D$ $E$ $F$ $G$

iii. $f'(x) < 0,\ f''(x) > 0$

$A$ $B$ $C$ $D$ $E$ $F$ $G$

13. After the consumption of an alcoholic beverage, the concentration of alcohol in the bloodstream surges as the alcohol is absorbed, followed by a gradual decline as the alcohol is metabolised. The function

$$C(t) = 1.2te^{-2.6t}$$

models the blood alcohol concentration, measured in mg/ml of a test patients bloodstream $t$ hours after rapidly consuming 15ml of alcohol.

(a) (4 marks) What is the blood alcohol concentration of the patient 1 hour after consumption?

$$C(1) = 1.2(1)e^{-2.6(1)} = \boxed{1.2e^{-2.6}\ \text{mg/ml}} \approx 0.089\ \text{mg/ml}$$

(b) (6 marks) Find $\frac{d}{dt}C(t)$.

Product rule

$f(t) = 1.2t \qquad g(t) = e^{-2.6t} \qquad c(t) = f(t)g(t)$

$f'(t) = 1.2 \qquad g'(t) = -2.6e^{-2.6t} \qquad c'(t) = f'(t)g(t) + g'(t)f(t)$

$$C'(t) = \boxed{1.2e^{-2.6t} - 3.12te^{-2.6t}}$$

(c) (6 marks) When does the blood alcohol concentration for this patient attain its maximum?

$$C'(t) = 1.2e^{-2.6t} - 3.12te^{-2.6t} = 0$$

$$\Longrightarrow 1.2e^{-2.6t} = 3.12te^{-2.6t}$$

$$\Longrightarrow 1.2 = 3.12t$$

$$\Longrightarrow \boxed{\frac{1.2}{3.12}} = t \qquad \approx 0.3846$$

(d) (4 marks) What is the maximum blood alcohol concentration for this patient?

$$C\left(\frac{1.2}{3.12}\right) = 1.2\left(\frac{1.2}{3.12}\right)e^{-2.6\left(\frac{1.2}{3.12}\right)} = \frac{1.2}{2.6}e^{-1} = \boxed{\frac{6}{13e}\ \text{mg/ml}}$$

$\approx 0.17$ mg/ml

14. At a price of \$6 per ticket, a musical theatre group can fill every seat in the theatre, which has a capacity of 2000. For every additional dollar charged, the number of people buying tickets decreases by 100.

(a) (5 marks) Find an equation relating the price of the ticket to the demand. That is, find an equation $q(p)$ that gives the quantity, $q$, of tickets sold in terms of the price, $p$. *(Hint: It's linear.)*

$$q - 2000 = -100(p-6)$$

$$q = 2000 - 100(p-6)$$

$$q = \boxed{2600 - 100p}$$

(b) (5 marks) Express the revenue generated by the theatre group as a function of *price*.

$$R = pq$$

$$R(p) = p(2600 - 100p) = \boxed{2600p - 100p^2}$$

(c) (10 marks) Find the price that the theatre company should sell each ticket for to maximise revenue.

$$R'(p) = 2600 - 200p = 0$$

$$2600 = 200p$$

$$\boxed{\$13 = p}$$

15. An industrial production process costs $C(q)$ million dollars to produce $q$ million units; these units then sell for $R(q)$ million dollars. If $C(2.1) = 5.1$, $R(2.1) = 6.9$, $MC(2.1) = 0.6$ and $MR(2.1) = 0.7$, calculate:

(a) (4 marks) The profit earned by producing 2.1 million units.

$$\pi(2.1) = R(2.1) - C(2.1) = 6.9 - 5.1 = \boxed{\$1.8 \text{ million}}$$

(b) (4 marks) The approximate change in revenue if the production increases from 2.1 to 2.4 million units.

$$(2.4 - 2.1)MR(2.1) = 0.3 \times 0.7 = \boxed{\$0.21 \text{ million}}$$

(c) (4 marks) The approximate change in cost if the production increases from 2.1 to 2.4 million units.

$$(2.4 - 2.1)MC(2.1) = 0.3 \times 0.6 = \boxed{\$0.18 \text{ million}}$$

(d) (4 marks) The approximate profit if the production increases from 2.1 to 2.4 million units.

$$\begin{aligned}\pi(2.4) &\approx \pi(2.1) + 0.3M\pi(2.1) \\ &= \pi(2.1) + 0.3MR(2.1) - 0.3MC(2.1) \\ &= 1.8 + 0.21 - 0.18 = \boxed{\$1.83 \text{ million}}\end{aligned}$$

(e) (4 marks) The approximate profit if the production decreases from 2.1 to 2.05 million units.

$$\begin{aligned}\pi(2.05) &\approx \pi(2.1) - 0.05M\pi(2.1) \\ &= \pi(2.1) - 0.05MR(2.1) + 0.05MC(2.1) \\ &= 1.8 - 0.05(0.7) + 0.05(0.6) \\ &= 1.8 - 0.035 + 0.03 \\ &= \boxed{\$1.795 \text{ million}}\end{aligned}$$